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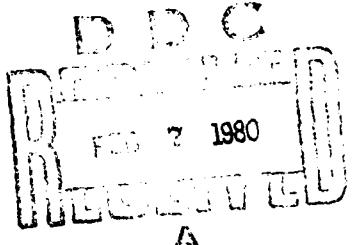
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ON MARKET GAMES

10) Lloyd S. Shapley — Martin Shubik

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PREFACE

This Memorandum extends to economic models some recently discovered counterexamples in n-person solution theory. It is part of a continuing investigation by the authors of the application of game theory to economics.

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SUMMARY

The "market games"—games that derive from an exchange economy in which the traders have continuous concave monetary utility functions, are shown to be the same as the "totally balanced games"—games which with all their subgames possess cores.

(The core of a game is the set of outcomes that no coalition can profitably block.) The coincidence of these two classes of games is established with the aid of explicit transformations that generate a game from a market and vice versa. It is further shown that any game with a core has the same solutions, in the von Neumann—Morgenstern sense, as some totally balanced game. Thus, a market may be found that reproduces the solution behavior of any game that has a core. In particular, using a recent result of Lucas (see *AD-661 544*), a ten-trader ten-commodity market is described that has no solution.

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## ON MARKET GAMES

### 1. INTRODUCTION

Recent discovery of n-person games in the classical theory which either possess no solutions [8, 9], or have unusually restricted classes of solutions [6, 7, 10, 18], has raised the question of whether these games are mere mathematical curiosities or whether they could actually arise in application. Since the most notable applications of n-person game theory to date have been to economic models of exchange, or exchange and production [3, 13, 15, 16, 19-23], the question may be put in a more concrete form: Are there markets, or other basic economic systems, that when interpreted as n-person games give rise to the newly-discovered counterexamples? If so, can they be distinguished from the ordinary run of market models, on some economic, heuristic, or even formal grounds—i.e., do they give any advance warning of their peculiar solution properties?

These questions stimulated the present investigation. The answers are "bad news": Yes, the games can arise in economics; No, there are no outwardly distinguishing features. In reaching these conclusions, however, we were led to a positive result: a surprisingly simple mathematical criterion that tells precisely which games can arise from economic models of exchange (with

money). In fact, this criterion identifies a very fundamental class of games, called "totally balanced", whose further study seems merited quite apart from any consideration of solution abnormalities. Of technical interest, our derivation of the basic properties of these games and their solutions makes a substantial application of the recently developed theory of balanced sets [1, 2, 11, 17], as well as of the older work of Gillies on domination-equivalence [4, 5].

In the present note we confine our attention to the classical theory "with side payments" [23]; this corresponds in the economic interpretation to the assumption that an ideal money, free from income effects or transfer costs, is available.\* We further restrict ourselves to exchange economies, without explicit production or consumption processes, in which the commodities are finite in number and perfectly divisible and transferable, and in which the traders, also finite in number, are motivated only by their own final holdings of goods and money, their utility functions being continuous and concave and additive in the money term.

For our immediate purpose, these strictures do not matter, since the anomalous games are already attainable within the limited class of ideal markets considered. But for our larger purpose—that of initiating a systematic study of "market games"

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\* For a discussion of this assumption, see [20], pp. 807-808.

as distinct from games in general—some relaxation may be desirable, particularly with regard to money. The prospects for significant generalizations in this direction appear good, and we intend to pursue them in subsequent work.

#### 1.1 Outline of the Contents

The notions of game, core, and balanced set are reviewed. A game is called "balanced" if it has a core, and "totally balanced" if all of the subgames obtained by restricting the set of players have cores as well. (Sec. 2.)

A "market" is defined as an exchange economy with money, in which the traders have utility functions that are continuous and concave. The method of passing from a market to its "market game" is described. The market games with  $n$  traders form a closed convex cone in the space of all  $n$ -person games with side payments. Every market game not only has a core, but is totally balanced. (Sec. 3.)

A canonical market form—the "direct market"—is introduced, in which the commodities are in effect the traders themselves, made infinitely divisible, and the utility functions are all the same and are homogeneous of degree one. The method of passing from any game to its direct market is described, the utilities being based upon the optimal assignment of "fractional players" to the various coalitional activities. The "cover" of a game is defined as the

market game of its direct market; the cover is at least as profitable to all coalitions as the original game. Every totally balanced game is its own cover, and hence is a market game. This shows that the class of market games and the class of totally balanced games are the same. Moreover, every market is game-theoretically equivalent to a direct market. (Sec. 4.)

The notions of imputation, domination, and solution are reviewed. Games are "d-equivalent" if they have identical domination relations on identical imputation spaces. They therefore have identical solutions (or lack of solutions), and their cores, if any, are the same. It is shown that every balanced game is d-equivalent to a totally balanced game. Hence, for every game with a core, there is a market that has precisely the same set of solutions. (Sec. 5.)

Using Lucas's solutionless game [8,9], a direct market is constructed that has ten traders and ten commodities (plus money), and that has no solution. Another version in the form of a production economy is also presented. Several other examples of market games with unusual solution properties are mentioned, and in one case, where the solutions contain arbitrary components, the utility function is worked out explicitly. (Sec. 6.)

## 2. GAMES AND CORES

For the purpose of this note, a game is an ordered pair  $(N; v)$ , where  $N$  is a finite set [the players] and  $v$  is a function from the subsets of  $N$  [coalitions] to the reals satisfying  $v(\emptyset) = 0$ , called the characteristic function. A payoff vector for  $(N; v)$  is a point  $\alpha$  in the  $|N|$ -dimensional vector space  $\mathbb{R}^N$  whose coordinates  $\alpha^i$  are indexed by the elements of  $N$ . If  $\alpha \in \mathbb{R}^N$  and  $S \subseteq N$ , we shall write  $\alpha(S)$  as an abbreviation for  $\sum_{i \in S} \alpha^i$ .

The core of  $(N; v)$  is the set of all payoff vectors  $\alpha$ , if any, such that

$$(2-1) \quad \alpha(S) \geq v(S), \quad \text{all } S \subseteq N,$$

and

$$(2-2) \quad \alpha(N) = v(N).$$

If no such  $\alpha$  exists, we shall say that  $(N; v)$  has no core. (Thus, in this usage, the core may be nonexistent, but is never empty.)

### 2.1 Balanced Sets of Coalitions

A balanced set  $\mathcal{B}$  is defined to be a collection of subsets  $S$  of  $N$  with the property that there exist positive numbers  $v_S$ ,  $S \in \mathcal{B}$ , called "weights," such that for each  $i \in N$  we have

$$(2-3) \quad \sum_{\substack{S \in \mathcal{B} \\ i \in S}} v_S = 1.$$

If all  $\gamma_S = 1$ , we have a partition of  $N$ ; thus, balanced sets may be regarded as generalized partitions.

For example, if  $N = \overline{1234}$ , then  $\{\overline{12}, \overline{13}, \overline{14}, \overline{234}\}$  is a balanced set, by virtue of the weights  $1/3, 1/3, 1/3, 2/3$ .

A game  $(N; v)$  is called balanced if

$$(2-4) \quad \sum_{S \in \mathcal{B}} \gamma_S v(S) \leq v(N)$$

holds for every balanced set  $\mathcal{B}$  with weights  $\{\gamma_S\}$ . \*

THEOREM 1. A game has a core if and only if it is balanced.

This is proved in [17]. In Scarf's generalization to games without transferable utility [11], all balanced games have cores, but some games with cores are not balanced. If our present results can be generalized in this direction, we conjecture that it will be the balance property, rather than the core property, that plays the central role.

## 2.2 Totally Balanced Games

By a subgame of  $(N; v)$  we shall mean a game  $(R; v)$  with  $O \subset R \subseteq N$ . Here  $v$  is the same function, but implicitly restricted to the domain consisting of the subsets of  $R$ . A game will be said

\* These conditions are heavily redundant; it suffices to assert (2-4) for the minimal balanced sets  $\mathcal{B}$  (which moreover have unique weights). In the case of a superadditive game, only the minimal balanced sets that contain no disjoint elements are needed. (See [17].)

to be totally balanced if all of its subgames are balanced. In other words, all subgames of a totally balanced game have cores.

Not all balanced games are totally balanced. For example, let  $N = \overline{1234}$  and define  $v(S) = 0, 0, 1, 2$  for  $|S| = 0, 1, 3, 4$  respectively, and, for  $|S| = 2$ :

$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 1$$

$$v(\overline{14}) = v(\overline{24}) = v(\overline{34}) = 0.$$

This game has a core, including the vector  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  among others. But it is not totally balanced, since the subgame  $(\overline{123}; v)$  has no core.

### 3. MARKETS AND MARKET GAMES

For the purpose of this note, a market is a special mathematical model, denoted by the symbol  $(T, G, A, U)$ . Here  $T$  is a finite set [the traders];  $G$  is the nonnegative orthant of a finite-dimensional vector space [the commodity space];  $A = \{a^i : i \in T\}$  is an indexed collection of points in  $G$  [the initial endowments]; and  $U = \{u^i : i \in T\}$  is an indexed collection of continuous, concave functions from  $G$  to the reals [the utility functions]. When we wish to indicate that  $u^i = u$ , all  $i \in T$  [the special case of "equal tastes"], we shall sometimes denote the market by the more specific symbol  $(T, G, A, \{u\})$ .

If  $S$  is any subset of  $T$ , an indexed collection  $X^S = \{x^i : i \in S\} \subset G$  such that  $\sum_{i \in S} x^i = \sum_{i \in S} a^i$  will be called a feasible  $S$ -allocation of the market  $(T, G, A, U)$ .

A market  $(T, G, A, U)$  can be used to "generate" a game  $(N; v)$  in a natural way. We set  $N = T$ , and define  $v$  by

$$(3-1) \quad v(S) = \max_{X^S} \sum_{i \in S} u^i(x^i), \quad \text{all } S \subseteq N,$$

where the maximum runs over all feasible  $S$ -allocations.

Any game that can be generated in this way from some market is called a market game.\*

\* For examples, see [13, 15, 16, 20, 21, 22]. The abstract definition of "market game" proposed in [13] is not equivalent to the present one, however.

In the special case of identical utility functions  $u^i = u$ , we have

$$(3-2) \quad v(S) = |S|u(\sum_S a^i / |S|), \quad \text{all } S \subseteq N;$$

this is a simple consequence of concavity. In the still more special case where  $u$  is homogeneous of degree 1, we have simply

$$(3-3) \quad v(S) = u(\sum_S a^i), \quad \text{all } S \subseteq N.$$

### 3.1 Some Elementary Properties

The following two theorems are of a routine nature; they show that the property of being a market game is invariant under "strategic equivalence," and that the set of all market games on  $N$  forms a convex cone in the  $(2^{|N|}-1)$ -dimensional space of all games on  $N$ .

THEOREM 2. If  $(N; v)$  is a market game, if  $\lambda \geq 0$ , and if  $c$  is an additive set function on  $N$ , then  $(N; \lambda v + c)$  is a market game.

Proof. We need merely take any market that generates  $(N; v)$  and replace each utility function  $u^i(x)$  by  $\lambda u^i(x) + c(\{i\})$ .

Q. E. D.

THEOREM 3. If  $(N; v')$  and  $(N; v'')$  are market games, then  $(N, v' + v'')$  is a market game.

Proof. Let  $(N, G', A', U')$  and  $(N, G'', A'', U'')$  be markets that generate  $(N; v')$  and  $(N; v'')$  respectively. We shall superimpose these two markets, keeping the two sets of commodities distinct. Specifically, let  $G$  be the set of all ordered pairs  $(x', x'')$  of points from  $G'$  and  $G''$  respectively; let  $A$  be the set of pairs  $(a'^i, a''^i)$  of correspondingly-indexed elements of  $A'$  and  $A''$ ; and let  $U$  be the set of sums:

$$u^i((x', x'')) = u'^i(x') + u''^i(x'')$$

of correspondingly-indexed elements of  $U'$  and  $U''$ . One can then verify without difficulty that the elements of  $U$  are continuous and concave on the domain  $G$  (which is a nonnegative orthant in its own right), so that  $(N, G, A, U)$  is a market. Finally, one can verify without difficulty that  $(N, G, A, U)$  generates the game  $(N; v'+v'')$ .

Q. E. D.

### 3.2 The Core Theorem

THEOREM 4. Every market game has a core.

This theorem is well-known, and has been generalized well beyond the limited class of markets we are now considering. Nevertheless we shall give two proofs, both short, for the sake of the insights they provide. In the first, we in effect determine a competitive equilibrium for the generating market (a simple matter

when there is transferable utility), and then show that the competitive payoff vector lies in the core. In the second proof, we show directly that the game is balanced, and then apply Theorem 1.

Proof 1. Let  $(N; v)$  be a market game and let  $(N, G, A, U)$  be a market that generates it. Let  $B = \{b^i : i \in N\}$  be a feasible  $N$ -allocation that achieves the value  $v(N)$  in (3-1) for  $S = N$ . The maximization in (3-1) ensures the existence of a vector  $p$  [competitive prices—but possibly negative!] such that for each  $i \in N$ , the expression

$$(3-4) \quad u^i(x^i) - p \cdot (x^i - a^i), \quad x^i \in G,$$

is maximized at  $x^i = b^i$ . Define the payoff vector  $\beta$  by  $\beta^i = u^i(b^i) - p \cdot (b^i - a^i)$ ; we assert that  $\beta$  is in the core. Indeed, let  $S$  be any nonempty subset of  $N$ , and let  $y^S$  be a feasible  $S$ -allocation that achieves the maximum in (3-1), so that  $v(S) = \sum_S u^i(y^i)$ . Since  $b^i$  maximizes (3-4), we have

$$\beta^i \geq u^i(y^i) - p \cdot (y^i - a^i).$$

Summing over  $i \in S$ , we obtain

$$\beta(S) \geq \sum_S u^i(y^i) - p \cdot 0 = v(S),$$

as required by (2-1). Moreover, if  $S = N$  we may take  $y^N = B$  and obtain  $\beta(N) = v(N)$ , as required by (2-2). Q. E. D.

Proof 2. Let  $(N, G, A, U)$  be a generating market for  $(N; v)$ , and, for each  $S \subseteq N$ , let  $Y_S^S = \{y_S^i : i \in S\}$  be a maximizing  $S$ -allocation in (3-1). Let  $\mathcal{B}$  be balanced, with weights  $\{\gamma_S : S \in \mathcal{B}\}$ . Then we have

$$\sum_{S \in \mathcal{B}} \gamma_S v(S) = \sum_{S \in \mathcal{B}} \sum_{i \in S} \gamma_S u^i(y_S^i) = \sum_{i \in N} \sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \gamma_S u^i(y_S^i).$$

Now define

$$z^i = \sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \gamma_S y_S^i \in G, \quad \text{all } i \in N.$$

Note that  $z^i$  is a center of gravity of the points  $y_S^i$ , by virtue of (2-3).

Hence, by concavity,

$$(3-5) \quad \sum_{S \in \mathcal{B}} \gamma_S v(S) \leq \sum_{i \in N} u^i(z^i).$$

But  $Z = \{z^i : i \in N\}$  is a feasible  $N$ -allocation, since

$$\sum_{i \in N} z^i = \sum_{S \in \mathcal{B}} \sum_{i \in S} \gamma_S y_S^i = \sum_{S \in \mathcal{B}} \gamma_S \sum_{i \in S} a^i = \sum_{i \in N} a^i.$$

Hence the right side of (3-5) is  $\leq v(N)$ , and we conclude from (2-4) that the game is balanced and from Theorem 1 that it has a core.

Q. E. D.

COROLLARY. Every market game is totally balanced.

Proof. If  $(N; v)$  is generated by the market  $(N, G, A, U)$ , and if  $O \subset R \subseteq N$ , then we may define a market  $(R, G, A', U')$ , where  $A'$  and  $U'$  come from  $A$  and  $U$  by simply omitting all  $a^i$  and  $u^i$  for  $i$  not in  $R$ . This market clearly generates the game  $(R; v)$ . Hence  $(R; v)$  is balanced. Q. E. D.

Our next objective will be to prove the converse of this corollary — i. e., that every totally balanced game is a market game.

#### 4. DIRECT MARKETS

A special class of markets, called direct markets, will play an important role in the sequel. They have the form

$$(T, E_+^T, I^T, \{u\}),$$

where  $u$  is homogeneous of degree 1 as well as concave and continuous. Here  $E_+^T$  denotes the nonnegative orthant of the vector space  $E^T$  with coordinates indexed by the members of  $T$ , and  $I^T$  denotes the collection of unit vectors of  $E^T$ —in effect, the identity matrix on  $T$ .

Thus, in a direct market, each trader starts with one unit of a personal commodity [e. g., his time, his labor, his participation, "himself"]. When it is brought together with other personal commodities, we may imagine that some desirable state of affairs is created, having a total value to the traders that is independent (because of homogeneity and equal tastes) of how they distribute the benefits.

Let  $e^S$  denote the vector in  $E^N$  in which  $e_i^S = 1$  or 0 according as  $i \in S$  or  $i \notin S$ ; geometrically, these vectors represent the vertices of the unit cube in  $E_+^N$ . Then the characteristic function of the market game generated by a direct market can be put into a very simple form:

$$(4-1) \quad v(S) = u(e^S), \quad \text{all } S \subseteq N$$

✓✓

(compare (3-3)). Note that only finitely many commodity bundles are involved in this expression.

#### 4.1 The Direct Market Generated by a Game

Thus far we used markets to generate games. We now go the reverse route, associating with any game (not necessarily a market game) a certain "market of coalitions." Specifically, we shall say that the game  $(N; v)$  "generates" the direct market  $(N, E_+^N, I^N, \{u\})$ , with  $u$  given by

$$(4-2) \quad u(x) = \max_{\{\gamma_S\}} \sum_{S \subseteq N} \gamma_S v(S), \quad \text{all } x \in E_+^N,$$

maximized over all sets of nonnegative  $\gamma_S$  satisfying

$$(4-3) \quad \sum_{S \ni i} \gamma_S = x_i, \quad \text{all } i \in N.$$

To explain this market, \* we may imagine that each coalition  $S$  has an activity  $\alpha_S$  that can earn  $v(S)$  dollars if all the members of  $S$  participate fully. More generally, it earns  $\gamma_S v(S)$  dollars if each member of  $S$  devotes the fraction  $\gamma_S$  of "himself" to  $\alpha_S$ . The maximization in (4-2) is then nothing but an optimal assignment of activity levels  $\gamma_S$  to the various  $\alpha_S$ 's, subject to the condition (4-3) that each player,  $i$ , distribute exactly the amount  $x_i$  of

\* The essence of this model was suggested by D. Cantor and M. Maschler (private correspondence, 1962).

"himself" among his activities, including of course the "solo" activity  $\alpha_{\{i\}}$ .

The utility function defined by (4-2) is obviously homogeneous of degree 1, as required for a direct market. But before we can claim to have defined a market, let alone a direct market, we must also establish that (4-2) is continuous and concave. Continuity gives no trouble. To show concavity, it suffices (with homogeneity) to prove that

$$u(x) + u(y) \leq u(x+y), \quad \text{all } x, y \in E_+^N.$$

This is not difficult. By definition, there exist sets of nonnegative coefficients  $\{\gamma_S\}$  and  $\{\delta_S\}$  such that

$$u(x) = \sum_{S \subseteq N} \gamma_S v(S), \quad u(y) = \sum_{S \subseteq N} \delta_S v(S);$$

and

$$\sum_{S \ni i} \gamma_S = x_i, \quad \sum_{S \ni i} \delta_S = y_i, \quad \text{all } i \in N.$$

Hence  $\{\gamma_S + \delta_S\}$  is admissible for  $x+y$ , and (4-2) yields

$$u(x+y) \geq \sum (\gamma_S + \delta_S) v(S) = u(x) + u(y),$$

as required.

#### 4.2. The Cover of a Game

We shall now use the direct market generated by a game  $(N; v)$  to generate in turn a new game  $(N; \bar{v})$ —schematically:

arbitrary game  $\longrightarrow$  direct market  $\longrightarrow$  market game.

We shall call  $(N; \bar{v})$  the cover of  $(N; v)$ .

Combining (4-1) with (4-2) and (4-3), we obtain the following relation between  $v$  and  $\bar{v}$ :

$$(4-4) \quad \bar{v}(R) = \max_{\{\gamma_S\}} \sum_{S \subseteq R} \gamma_S v(S), \quad \text{all } R \subseteq N$$

maximized over  $\gamma_S \geq 0$  such that

$$(4-5) \quad \sum_{\substack{S \subseteq R \\ S \ni i}} \gamma_S = 1, \quad \text{all } i \in R.$$

Note that we could have taken (4-4), (4-5) as the definition of "cover", bypassing the intermediate market. Indeed, the cover of a game proves to be a useful mathematical concept quite apart from the present economic application.

We see immediately that

$$(4-6) \quad \bar{v}(R) \geq v(R), \quad \text{all } R \subseteq N,$$

since one of the admissible choices for  $\{\gamma_S\}$  in (4-4) is to take  $\gamma_R = 1$  and all other  $\gamma_S = 0$ . Moreover, the equality cannot always hold in (4-6); indeed,  $\bar{v}$  comes from a market game while  $v$  was

arbitrary. Thus, the mapping  $v \rightarrow \bar{v}$  takes an arbitrary characteristic function and, by perhaps increasing some values, turns it into the characteristic function of a market game.

LEMMA 1. If  $(N; v)$  has a core, then  $\bar{v}(N) = v(N)$ ,  
and conversely.

Proof. Let  $\alpha$  be in the core of  $(N; v)$ . Then

$$\begin{aligned}\bar{v}(N) &= \max_{\{\gamma_S\}} \sum_{S \subseteq N} \gamma_S v(S) \\ &\leq \max_{\{\gamma_S\}} \sum_{S \subseteq N} \gamma_S \alpha(S) = \max_{\{\gamma_S\}} \sum_{i \in N} \alpha^i \sum_{S \ni i} \gamma_S \\ &= \max_{\{\gamma_S\}} \sum_{i \in N} \alpha^i = \alpha(N) \\ &= v(N),\end{aligned}$$

the successive lines being justified by (4-4), (2-1), (4-5), and (2-2). In view of (4-6) we therefore have  $\bar{v}(N) = v(N)$ .

Conversely, if  $(N; v)$  has no core, then (2-4) fails for some balanced set  $\mathcal{B}$  with weights  $\{\gamma_S\}$ . Defining  $\gamma_S = 0$  for  $S \notin \mathcal{B}$ , we see that (4-5) holds (for  $R = N$ ). Then (4-4) and the denial of (2-4) give us

$$\bar{v}(N) \geq \sum_{S \subseteq N} \gamma_S v(S) = \sum_{S \in \mathcal{B}} \gamma_S v(S) > v(N);$$

Hence  $\bar{v}(N) \neq v(N)$ . Q. E. D.

LEMMA 2. A totally balanced game is equal to its cover.

Proof. Let  $(N; \bar{v})$  be the cover of  $(N; v)$ , and let  $0 \subset R \subseteq N$ . Then it is clear from the definitions that the cover of  $(R; v)$  is  $(R; \bar{v})$ . But if  $(N; v)$  is totally balanced, then  $(R; v)$  has a core and  $\bar{v}(R) = v(R)$  by Lemma 1. Hence  $\bar{v} = v$ . Q. E. D.

THEOREM 5. A game is a market game if and only if it is totally balanced.

Proof. We proved earlier (corollary to Theorem 4) that market games are totally balanced. We have just now shown that totally balanced games are equal to their covers, which are market games. Q. E. D.

#### 4.3 Equivalence of Markets

There is one more result of some heuristic interest that we can extract from the present discussion, before entering the realm of solution theory. This time, we follow the scheme:

arbitrary market  $\longrightarrow$  market game  $\longrightarrow$  direct market.

Let us call two markets game-theoretically equivalent if they generate the same market game. Then the two markets in the above scheme are equivalent in this way, since the cover of the game in the middle is just the market game of the market on the

right, and these two games are equal by Lemma 2. This proves

THEOREM 6. Every market is game-theoretically  
equivalent to a direct market.

## 5. SOLUTIONS

An imputation for a game  $(N; v)$  is a payoff vector  $\alpha$  that satisfies

$$(5-1) \quad \alpha(N) = v(N)$$

and

$$(5-2) \quad \alpha^i \geq v(\{i\}), \quad \text{all } i \in N.$$

A comparison with (2-1) and (2-2) shows that the imputation set is certainly not empty if the game has a core.\*

Classical solution theory [23] rests upon a relation of "domination" between imputations. If  $\alpha$  and  $\beta$  are imputations for  $(N; v)$ , then  $\alpha$  is said to dominate  $\beta$  ( $v$ ritten  $\alpha \rightarrow \beta$ ) if there is some nonempty subset  $S$  of  $N$  such that

$$(5-3) \quad \alpha^i > \beta^i, \quad \text{all } i \in S,$$

and

$$(5-4) \quad \alpha(S) \leq v(S).$$

---

\* Some approaches to solution theory omit (5-2), relying on the solution concept itself to impose whatever "individual rationality" the situation may demand [5, 12]. This modification in the definition of solution would make little difference to our present discussion, except for eliminating the fussy condition (5-5). In particular, Theorem 7 and all of Sec. 6 would remain correct as written.

A solution of  $(N; v)$  is defined to be any set of imputations, mutually undominated, that collectively dominate all other imputations. Our only concern with this definition, technically, is to observe that it depends only on the concepts of imputation and domination; any further information conveyed by the characteristic function is disregarded.

The core is also closely dependent on these concepts. In fact, the core, when it exists, is precisely the set of undominated imputations. The converse is not universally true—there are some games that have undominated imputations but no core.\* We can rule this out, however, by imposing the very weak condition:

$$(5-5) \quad v(S) + \sum_{N-S} v(\bar{i}) \leq v(N), \quad \text{all } S \subseteq N,$$

which is satisfied by all games likely to be met in practice.\*\*

### 5.1 Domination-equivalence

Two games will be called d-equivalent (domination-equivalent) if they have the same imputation sets and the same domination relations on them. It follows that d-equivalent games have

\* We are indebted to Mr. E. Kohlberg for this observation.

\*\* Thus, (5-5) is implied by either superadditivity or balancedness, but is weaker than both. For a game in normalized form, i.e., with  $v(\bar{i}) = 0$ , it merely states that no coalition is worth more than  $N$ .

precisely the same solutions, or lack of solutions. Also, if they have cores, they have the same cores; moreover, within the class of games satisfying (5-5) the property of being balanced is preserved under d-equivalence. However, the property of being totally balanced is not so preserved, as the following lemma reveals.

LEMMA 3. Every balanced game is d-equivalent to its cover.

Proof. Let  $(N; v)$  be balanced. By Lemma 1,  $\bar{v}(N) = v(N)$  and by (4-4),  $\bar{v}(\{i\}) = v(\{i\})$ ; hence the two games have the same imputations. Denote the respective domination relations by  $\leftarrow$  and  $\leftarrow'$ . By (4-6) and (5-4) we see at once that the latter is, if anything, stronger than the former—i.e.,  $\alpha \leftarrow \beta$  implies  $\alpha \leftarrow' \beta$ . It remains to prove the converse.

Assume, per contra, that  $\alpha$  and  $\beta$  are imputations satisfying  $\alpha \leftarrow' \beta$  but not  $\alpha \leftarrow \beta$ . Then for some nonempty subset  $R$  of  $N$  we have

$$\alpha^i > \beta^i, \quad \text{all } i \in R,$$

and

$$(5-6) \quad \alpha(R) \leq \bar{v}(R).$$

To avoid  $\alpha \leftarrow \beta$  we must have

$$(5-7) \quad \alpha(S) > v(S)$$

for all  $S$ ,  $0 \subset S \subseteq R$ . Referring to the definition of  $\bar{v}$ , we see that there are nonnegative weights  $\gamma_S$ ,  $S \subseteq R$ , such that

$$\bar{v}(R) = \sum_{S \subseteq R} \gamma_S v(S)$$

and

$$\sum_{\substack{S \subseteq R \\ S \ni i}} \gamma_S = 1, \quad \text{all } i \in R.$$

Hence, using (5-7)

$$\bar{v}(R) < \sum_{S \subseteq R} \gamma_S \alpha(S) = \alpha(R).$$

The strict inequality here contradicts (5-6). Q. E. D.

By a "solution" of a market, we shall mean a solution of the associated market game.

**THEOREM 7.** If  $(N; v)$  is any balanced game whatever,  
then there is a market that has precisely the same  
solutions as  $(N; v)$ .

**Proof.** The main work has been done in Lemma 3. Indeed, let  $(N, E_+^T, I^T, \{u\})$  be the direct market generated by  $(N; v)$ .

Then the solutions of this market are the solutions of  $(N; \bar{v})$ , which by the lemma is d-equivalent to  $(N; v)$  and hence has the same solutions. Q. E. D.

### 5.2 A Technical Remark

The notion of d-equivalence is essentially due to Gillies [4, 5], though he works with a broader definition of imputation, not tied to the characteristic function by (5-1). He defines a vital coalition as one that achieves some domination that no other coalition can achieve, and shows that two games are d-equivalent (in the present sense) if and only if they have (i) the same imputation sets, (ii) the same vital coalitions, and (iii) the same v-values on their vital coalitions.

A necessary (but not sufficient) condition for a coalition to be vital is that it cannot be partitioned into proper subsets, the sum of whose v-values equals or exceeds its own v-value. Sufficiency would require the generalized partitioning provided by balanced sets.

Given a game  $(N; v)$ , we can define its "least superadditive majorant"  $(N; \tilde{v})$  by

$$(5-8) \quad \tilde{v}(S) = \max \sum_h v(S_h),$$

the maximization running over all partitions  $\{S_h\}$  of  $S$ . (Compare (4-4), (4-5).) It can be shown that  $\tilde{v}(N) = v(N)$  if and only if  $(N; v)$  has a core (cf. Lemma 1 above), in which case the two games are d-equivalent. Thus, every game with a core is d-equivalent to a superadditive game.

However, as Gillies observes, d-equivalence can also hold nontrivially among superadditive games. That is, it may be possible to push the  $v$ -value of some nonvital coalition higher than the value demanded by superadditivity, without making the coalition vital.\* We are using the full power of this observation, since the cover  $\bar{v}$  can be thought of as the "greatest d-equivalent majorant" of  $v$ . Thus,  $v \leq \tilde{v} \leq \bar{v}$ , and all three may be different.

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\* For example, at the end of Sec. 2,  $v(\bar{123})$  may be increased from 1 to  $3/2$  without making 123 vital.

## 6. EXAMPLES

Lucas's 10-person game [8, 9] with no solution has players

$N = \overline{1234567890}$  and the following characteristic function:

$$(6-1) \quad \left\{ \begin{array}{l} v(\overline{12}) = v(\overline{34}) = v(\overline{56}) = v(\overline{78}) = v(\overline{90}) = 1 \\ v(\overline{137}) = v(\overline{139}) = v(\overline{157}) = v(\overline{159}) = v(\overline{357}) = v(\overline{359}) = 2 \\ v(\overline{1479}) = v(\overline{2579}) = v(\overline{3679}) = 2 \\ v(\overline{1379}) = v(\overline{1579}) = v(\overline{3579}) = 3 \\ v(\overline{13579}) = 4 \\ v(N) = 5, \text{ and} \\ v(S) = 0, \text{ all other } S \subseteq N. \end{array} \right.$$

The game has a core, containing among others the imputation that gives each "odd" player 1. \* It is not superadditive (for example,  $v(\overline{12}) + v(\overline{34}) > v(\overline{1234})$ ); however it is d-equivalent to its least superadditive majorant  $(N, \tilde{v})$ , which can be calculated without difficulty, using (5-8). Moreover, one can verify that the latter is totally balanced, i.e., that  $\tilde{v} = \bar{v}$  in this case. Thus,  $(N, \tilde{v})$ , defined by (6-1) and (5-8), is a market game with no solution.

The corresponding market with no solution, provided by Theorem 7, has ten traders and ten commodities, plus money. The traders have identical continuous concave homogeneous utilities

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\* The full core is a five-dimensional polyhedron, having vertices  $e^S: S = \overline{13579}, \overline{23579}, \overline{14579}, \overline{13679}, \overline{13589}, \text{ and } \overline{13570}$ .

$u(x)$ , which may be calculated by applying (4-2), (4-3) to (6-1).

Note that positive weights  $\gamma_S$  need be considered only for the eighteen vital coalitions and the ten singletons.\* Of course, this is not the only utility function that works, since only a finite set of its values are actually used (cf. (4-1)).

### 6.1 A Production Model

Perhaps the most straightforward economic realization of Lucas's game is in the form of a production economy. (Compare the "activity" description in Sec. 4.1.) The production possibilities are generated by 18 specific processes, which produce the same consumer good (at constant returns to scale) out of various combinations of the raw materials. (See Table 1.) Each entrepreneur starts with one unit of the correspondingly indexed raw material. The utility is simply the consumer good:  $u(x) : x_{11}$ ; hence it is not necessary to postulate a separate money.

This type of construction is perfectly general: a production model can be set up in a similar fashion for any other game in characteristic function form, one activity being required for each vital coalition. The market game generated by such a model will

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\* The singleton weights are needed as slack variables, because we used "=" in (4-3) instead of " $\leq$ ".

<u>Inputs</u>										<u>Output</u>
$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$x_9$	$x_{10}$	$x_{11}$
1	1									1
		1	1							1
				1	1					1
						1	1			1
								1	1	1
1		1					1			2
1		1						1		2
1			1				1			2
1			1					1		2
		1	1				1			2
		1	1					1		2
1			1				1	1		2
	1			1			1	1		2
		1			1		1	1		2
1		1					1	1		3
1			1				1	1		3
		1	1				1	1		3
1	1	1					1	1		4

Table 1.

be the cover of the original game, and will have the same core and solutions provided that the original game was balanced.

### 6.2 Other Examples

Lucas [6, 7, 10] gives several examples of games in which the solution is unique but does not coincide with the core. In [10] he also describes a symmetric 8-person game, very similar to the above 10-person game, that has an infinity of solutions but none that treats the symmetric players symmetrically. Shapley [18] describes a 20-person game, of the same general type, every one of whose many solutions consists of the core, which is a straight line, plus an infinity of mutually disjoint closed sets that intersect the core in a dense point-set of the first category. A common feature of all these "pathological" examples is the existence of a core; hence, by Theorem 7, they are d-equivalent to market games that have the same solution behavior.

We close with another "pathological" example, of an older vintage [14], which because of its simple form leads to a direct market with utilities that we can write down explicitly. The game has players  $N = \overline{123 \dots n}$ , with  $n \geq 4$ , and its characteristic function is given by

$$(6-2) \quad \begin{cases} v(N-\{1\}) = v(N-\{2\}) = v(N-\{3\}) = v(N) = 1 \\ v(S) = 0, \quad \text{all other coalitions } S. \end{cases}$$

Thus, to win anything requires the participation of a majority of  $\overline{123}$ , plus all of the "veto" players  $4, \dots, n$ . The core is the set of all imputations  $\alpha$  that satisfy  $\alpha_1 = \alpha_2 = \alpha_3 = 0$ . It is easily verified that the game is totally balanced:  $v = \bar{v}$ . There are many solutions; but the remarkable feature of the game is a certain subclass of solutions, as follows:

Let  $B_e$  denote the set of imputation  $\alpha$  that satisfy  $\alpha_1 = 0$ ,  $\alpha_2 = \alpha_3 = e > 0$ . Thus,  $B_e$  is a  $(n-3)$ -dimensional closed convex subset of the imputation space. In [14] it was shown that one may start with any closed subset of  $B_e$  whatever, and extend it to a solution of the game by adding only imputations that are at least  $e/2$  distant from  $B_e$ .\* The arbitrary starting set remains a distinct, isolated portion of the full solution. For example, if  $n = 4$  (the simplest case), an arbitrary closed set of points on a certain line can be used.

To determine the direct market of this game, we apply (4-2), (4-3) to (6-2) and obtain the utility function

\* The metric used here is  $\rho(\alpha, \beta) = \max_i |\alpha_i - \beta_i|$ . Our present claim entails a slight change in the construction given in [14], which merely keeps the rest of the solution away from the arbitrary subset of  $B_e$ , rather than from  $B_e$  itself.

$$u(x) = \max_{\{y^i\}} (y^1 + y^2 + y^3),$$

maximized subject to

$$\left\{ \begin{array}{l} y^1 \geq 0, y^2 \geq 0, y^3 \geq 0; \\ y^2 + y^3 \leq x_1, y^1 + y^3 \leq x_2, y^1 + y^2 \leq x_3; \\ y^1 + y^2 + y^3 \leq x_i, i = 4, \dots, n; \end{array} \right.$$

where  $y^i$  abbreviates  $y_{N-\{i\}}$ . This reduces to the closed form:

$$(6-3) \quad u(x) = \min \left[ x_1 + x_2, x_1 + x_3, x_2 + x_3, \frac{x_1 + x_2 + x_3}{2}, x_4, \dots, x_n \right].$$

We see that  $u$  is the envelope-from-below of  $n+1$  very simple linear functions.

Thus, an  $n$ -trader  $n$ -commodity market having the solutions containing arbitrary components, as described above, is obtained by giving the  $i$ -th trader one unit of the  $i$ -th commodity,  $i = 1, \dots, n$ , and assigning them all the utility function (6-3).

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